Notes on effective waves in a multi-species material

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Abstract

This Supplementary Material is a self-contained document providing further detail on the calculation of effective wavenumbers for uniformly distributed multi-species inclusions. The formulae for multi-species cylinders and spheres are given here, in addition to expressions describing reflection from a halfspace filled with cylinders. Code to implement the formulas is given in `github.com/arturgower/EffectiveWaves.jl`. For detailed derivations see our paper (A. L. Gower et al., 2017), which shows how to introduce a pair-correlation between the species.

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1 Effective waves for uniformly distributed species

We consider a halfspace $x > 0$ filled with S types of inclusions (species) that are uniformly distributed. The fields are governed by the scalar wave equation:

$$\nabla^2 u + k^2 u = 0, \quad \text{(in the background material)} \tag{1}$$

$$\nabla^2 u + k_j^2 u = 0, \quad \text{(inside the } j\text{-th scatterer)}, \tag{2}$$

The background and species material properties are summarised in Table 1. The goal is to find an effective homogeneous medium with wavenumber k_*, where waves propagate, in an ensemble average sense, with the same speed and attenuation as they would in a material filled with scatterers. See A. Gower (2017) for the code that implements the formulas below.

Below we present the effective wavenumber, for any incident wavenumber and moderate number fraction, when the species are either all cylinders or spheres*. For cylindrical inclusions we also present the reflection of a plane wave from this multi-species material.

Background properties:	wavenumber k	density ρ	sound speed c	
Specie properties:	number density $\mathfrak{n}_j$	density ρ_j	sound speed c_j	radius a_j
total number density $\mathfrak{n}$	effective wavenumber k_*	species min. distance $a_{j\ell} > a_j + a_\ell$		

Table 1: Summary of material properties and notation. The index j refers to properties of the j-th species. Note a typical choice for $a_{j\ell}$ is $a_{j\ell} = c(a_j + a_\ell)$, where $c = 1.01$.

*In principal these formulas can be extended to include different shaped scatterers by using Waterman's T-matrix Waterman (1971); Varadan et al. (1978); Mishchenko et al. (1996)

2 Cylindrical species

We consider an incident wave

$$u_{\text{in}} = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \quad \text{with} \quad \mathbf{k}\cdot\mathbf{x} = kx\cos\theta_{\text{in}} + ky\sin\theta_{\text{in}}, \tag{3}$$

and angle of incidence θ_{in} from the x-axis, exciting a material occupying the halfspace $x > 0$. Then, assuming low number density $\mathfrak{n}$ (or low volume fraction $\sum_\ell \pi a_\ell^2 \mathfrak{n}_\ell$), the effective transmitted wavenumber k_* becomes

$$k_*^2 = k^2 - 4\mathrm{i}\mathfrak{n}\langle f_\circ\rangle(0) - 4\mathrm{i}\mathfrak{n}^2\langle f_{\circ\circ}\rangle(0) + \mathcal{O}(\mathfrak{n}^3), \tag{4}$$

with $\langle f_\circ\rangle$ and $\langle f_{\circ\circ}\rangle$ given by (8). The above reduces to Linton and Martin (2005) equation (81) for a single species in the low frequency limit; This equation (81) has been confirmed by several independent methods Martin et al. (2010); Martin and Maurel (2008); Chekroun et al. (2012); Kim (2010).

The ensemble-average reflected wave measured at $x < 0$ is given by

$$\langle u_{\text{ref}}\rangle = \frac{\mathfrak{n}}{\alpha^2}\left[R_1 + \mathfrak{n}R_2\right]\mathrm{e}^{-\mathrm{i}\alpha x + \mathrm{i}\beta y} + \mathcal{O}(\mathfrak{n}^3), \tag{5}$$

where

$$R_1 = \mathrm{i}\langle f_\circ\rangle(\theta_{\text{ref}}), \quad \theta_{\text{ref}} = \pi - 2\theta_{\text{in}}, \tag{6}$$

$$R_2 = \frac{2\langle f_\circ\rangle(0)}{k^2\cos^2\theta_{\text{in}}}\left[\sin\theta_{\text{in}}\cos\theta_{\text{in}}\langle f_\circ\rangle'(\theta_{\text{ref}}) - \langle f_\circ\rangle(\theta_{\text{ref}})\right] + \mathrm{i}\langle f_{\circ\circ}\rangle(\theta_{\text{ref}}), \tag{7}$$

which reduces to Martin (2011) equations (40-41) for a single species, which they show agrees with other known results for small k.

The ensemble-average far-field pattern and multiple-scattering pattern are[†]

$$\langle f_\circ \rangle(\theta) = -\sum_{\ell=1}^{S} \sum_{n=-\infty}^{\infty} \frac{\mathfrak{n}_\ell}{\mathfrak{n}} Z_\ell^n \mathrm{e}^{\mathrm{i}n\theta},$$

$$\langle f_{\circ\circ} \rangle(\theta) = -\pi \sum_{\ell,j=1}^{S} \sum_{m,n=-\infty}^{\infty} a_{\ell j}^2 d_{n-m}(k a_{\ell j}) \frac{\mathfrak{n}_\ell \mathfrak{n}_j}{\mathfrak{n}^2} Z_\ell^n Z_j^m \mathrm{e}^{\mathrm{i}n\theta}, \tag{8}$$

where $d_m(x) = J'_m(x)H'_m(x) + (1 - (m/x)^2)J_m(x)H_m(x)$, the J_m are Bessel functions, the H_m are Hankel functions of the first kind and $a_{\ell j} > a_\ell + a_j$ is some fixed distance. The Z_j^m describe the type of scatterer:

$$Z_j^m = \frac{q_j J'_m(k a_j) J_m(k_j a_j) - J_m(k a_j) J'_m(k_j a_j)}{q_j H'_m(k a_j) J_m(k_j a_j) - H_m(k a_j) J'_m(k_j a_j)} = Z_j^{-m}, \tag{9}$$

with $q = (\rho_j c_j)/(\rho c)$. For instance, taking the limits $q \to 0$ or $q \to \infty$, recovers Dirichlet or Neumann boundary conditions, respectively.

2.1 Any volume fraction

The series expansions for low number density (or volume fraction) do not work when the particles are strong scatterers. In these cases we need to use formulas valid for any volume fraction.

[†]Note we introduced the terminology "multiple-scattering pattern".

Borrowing equations (45 - 47) from A. L. Gower et al. (2017) we have

$$k_* \sin\theta_* = k \sin\theta_{\text{in}} \quad \text{with} \quad \mathbf{k}_* = (\alpha_*, \beta) := k_*(\cos\theta_*, \sin\theta_*), \tag{10}$$

$$\sum_\ell \sum_{n=-\infty}^{\infty} (2\pi \mathfrak{n}_\ell \mathcal{Q}_{j\ell}^{n-m}(k_*) Z_\ell^n + \delta_{mn}\delta_{j\ell}) \mathcal{A}_\ell^n = 0, \tag{11}$$

$$2 \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n(\theta_{\text{in}}-\theta_*)} \sum_\ell \mathfrak{n}_\ell Z_\ell^n \mathcal{A}_\ell^n = (\alpha_* - \alpha)\mathrm{i}\alpha, \tag{12}$$

in terms of the unknown parameters $\mathcal{A}_\ell^n$ and k_*, where

$$\mathcal{Q}_{j\ell}^n(k_*) = \frac{\mathcal{N}_n(k a_{j\ell}, k_* a_{j\ell})}{k^2 - k_*^2} + \mathcal{X}_n(\mathbf{s}_j, \mathbf{s}_\ell), \tag{13}$$

$$\mathcal{N}_n(x, y) = x H_n'(x) J_n(y) - y H_n(x) J_n'(y), \tag{14}$$

and $\mathcal{X}_n = 0$ for *hole correction*, or for a more general pair distribution

$$\mathcal{X}_n(\mathbf{s}_j, \mathbf{s}_\ell) = \int_{a_{j\ell} < R < \bar{a}_{j\ell}} H_n(kR) J_n(k_* R) \chi(R|\mathbf{s}_j, \mathbf{s}_\ell) R \, dR, \tag{15}$$

where we assume that when the distance between two cylinders $R_{j\ell} > \bar{a}_{j\ell}$, then the pair correlation is the same as hole correction.

In the notation given in A. L. Gower et al. (2017) we replaced $\mathcal{A}_*^m(\mathbf{s}_2) \to \mathcal{A}_\ell^m$, $p(\mathbf{s}_2) \to \delta(\mathbf{s}_2 - \mathbf{s}_\ell)\frac{\mathfrak{n}_\ell}{\mathfrak{n}}$, $\mathfrak{n} = \sum_\ell \mathfrak{n}_j$, $\mathfrak{n} = \sum_\ell \mathfrak{n}_j$, $Z^n(\mathbf{s}_2) \to Z_j^n$, $\mathcal{X}_* \to \mathcal{X}_{n-m}(\mathbf{s}_j, \mathbf{s}_\ell)$ and here we assumed no boundary-layer $\bar{x} = 0$.

Now we approximate (12,11) by summing n from $-N$ to N and then rewriting these equations as

$$\sum_\ell \boldsymbol{M}_{j\ell} \boldsymbol{A}_\ell = 0, \implies \det(\boldsymbol{M}_{j\ell}) = 0, \tag{16}$$

where

$$(\boldsymbol{A}_\ell)_n = \mathcal{A}_\ell^n, \quad (\boldsymbol{M}_{j\ell})_{mn} = 2\pi\mathfrak{n}_\ell Z_\ell^n \mathcal{Q}_{j\ell}^{n-m}(k_*) + \delta_{mn}\delta_{j\ell}, \tag{17}$$

and $n, m = -N, -N+1, \ldots N$.

The strategy to solve these equations is to: find k_* such that the determnant in (16) is zero and then find the eigenvector $\mathbb{A}$ of $(\boldsymbol{M}_{j\ell})$ with the smallest eignvalue; use Snell's law (10) to find θ_*; finally use (12) to determine the magnitude of $\mathbb{A}$.

One concern, is that the solutions k_* to (16) are not unique.

2.1.1 Reflection coefficient

Borrowing equation (88) from A. L. Gower et al. (2017), the average reflection from a halfspace is

$$\langle u_{\text{ref}}(x,y)\rangle = \mathrm{e}^{\mathrm{i}k(-x\cos\theta_{\text{in}}+y\sin\theta_{\text{in}})}R, \tag{18}$$

$$R = \frac{2\mathrm{i}}{\alpha(\alpha+\alpha_*)}\sum_{\ell=1}^{S}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n\theta_{\text{ref}}}\mathfrak{n}_\ell\mathcal{A}_\ell^n Z_\ell^n, \tag{19}$$

with $\theta_{\text{ref}} = \pi - \theta_{\text{in}} - \theta_*$.

3 Spherical species

The results here are derived by applying the theory in our paper to the results in Linton and Martin (2006). We omit the details as the result follows by direct analogy.

For spherical inclusions the transmitted wavenumber becomes,

$$k_*^2 = k^2 - \mathfrak{n}\frac{4\pi \mathrm{i}}{k}\langle F_\circ\rangle(0) + \mathfrak{n}^2\frac{(4\pi)^2}{k^4}\langle F_{\circ\circ}\rangle + \mathcal{O}(\mathfrak{n}^3), \tag{20}$$

where for spheres we define the ensemble-average far-field pattern and multiple-scattering pattern,

$$\langle F_\circ\rangle(\theta) = -\sum_{n=0}^{\infty} P_n(\cos\theta)\sum_{j=1}^{S}(2n+1)\zeta_j^n\frac{\mathfrak{n}_j}{\mathfrak{n}}, \tag{21}$$

$$\langle F_{\circ\circ}\rangle = \frac{\mathrm{i}(4\pi)^2}{2}\sum_{n,p=0}^{\infty}\sum_{j,\ell=1}^{S}\sum_{q}\frac{\sqrt{(2n+1)(2p+1)}}{(4\pi)^{3/2}}\sqrt{2q+1}\mathcal{G}(n,p,q)ka_{j\ell}D_q(ka_{j\ell})\zeta_j^n\zeta_\ell^p\frac{\mathfrak{n}_j\mathfrak{n}_\ell}{\mathfrak{n}^2},$$

where

$$D_m(x) = xj_m'(x)(xh_m'(x) + h_m(x)) + (x^2 - m(m+1))j_m(x)j_m(x),$$

P_n are Legendre polynomials, j_m are spherical Bessel functions, h_m are spherical Hankel functions of the first kind and

$$\zeta_j^m = \frac{q_j j_m'(ka_j)j_m(k_ja_j) - j_m(ka_j)j_m'(k_ja_j)}{q_j h_m'(ka_j)j_m(k_ja_j) - h_m(ka_j)j_m'(k_ja_j)} = \zeta_j^{-m}, \tag{22}$$

with $q = (\rho_j c_j)/(\rho c)$, where the $\mathcal{G}$ is a Gaunt coefficient and is equal to

$$\mathcal{G}(n,p,q) = \frac{\sqrt{(2n+1)(2p+1)(2q+1)}}{2\sqrt{4\pi}}\int_0^\pi P_n(\cos\theta)P_p(\cos\theta)P_q(\cos\theta)\sin\theta d\theta.$$

See Caleap et al. (2012) for details on reflection from a single species, although, to our knowledge, a formula for reflection from a single species valid for moderate number fraction and any wavenumber has not yet been deduced.

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