Acoustic multiple scattering

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January 7, 2021

Abstract

Here we show and deduce the T-matrix and multiple scattering for acoustics. Before reading this document, it may be helpful to read the general multiple scattering formulation shown in multiplescattering.pdf.

Keywords: Multiple scattering, T-matrix, Scattering matrix

1 2D acoustics

Much of the notation is define in multiplescattering.pdf. For the 2D acoustics some good references are [2, 1].

For 2D acoustics we have that

$$\mathrm{u}_n(k\boldsymbol{r}) = J_n(kr)\mathrm{e}^{\mathrm{i}n\theta}, \tag{1}$$

$$\mathrm{v}_n(k\boldsymbol{r}) = H_n(kr)\mathrm{e}^{\mathrm{i}n\theta}. \tag{2}$$

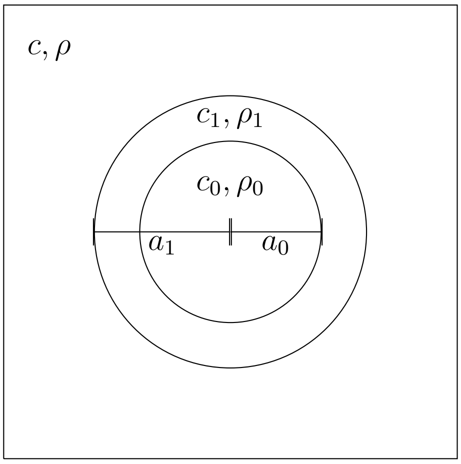
When truncating up to some order N we would sum over $n = -N, -N + 1, \ldots, N-1, N$.

1.1 Circular cylinder

Let ρ and c be the background density and wavespeed, and let ρ_j, c_j and radius a_j be the mass density, wavespeed, and radius for a circular scatterer with density.

Let $u = u_{\mathrm{inc}} + u_{\mathrm{sc}}$ be the total field outside the particle, and v_{in} the total field inside the particle, then from the acoustic boundary conditions:

$$u = v_{\mathrm{in}}, \quad \frac{1}{\rho}\frac{\partial u}{\partial r} = \frac{1}{\rho_j}\frac{\partial v_{\mathrm{in}}}{\partial r}, \quad \text{for} \ \ r = a_j,$$



we can deduce the T-matrix

$$T_{nm} = -\delta_{nm} \frac{\gamma_j J'_m(ka_j) J_m(k_j a_j) - J_m(ka_j) J'_m(k_j a_j)}{\gamma_j H'_m(ka_j) J_m(k_j a_j) - H_m(ka_j) J'_m(k_j a_j)}, \tag{3}$$

where $\gamma_j = (\rho_j c_j)/(\rho c)$ and $k_j = \omega/c_j$.

We can also calculate the coefficients b_n from

$$b_n = \frac{f_n}{T_{nn} J_n(k_j a_j)} \left[T_{nn} H_n(ka_j) + J_n(ka_j) \right] \tag{4}$$

1.2 Circular cylindrical capsule

$$\psi^0 = \sum_{n=-\infty}^{\infty} g_n^0 J_n(k_0 r) \mathrm{e}^{\mathrm{i}n\theta}, \tag{5}$$

$$\psi^1 = \sum_{n=-\infty}^{\infty} \left[g_n^1 J_n(k_1 r) + f_n^1 H_n(k_1 r) \right] \mathrm{e}^{\mathrm{i}n\theta}. \tag{6}$$

Applying the boundary conditions,

$$\psi^0 = \psi^1 \quad \text{and} \quad \frac{1}{\rho_0} \frac{\partial \psi^0}{\partial r} = \frac{1}{\rho_1} \frac{\partial \psi^1}{\partial r}, \quad \text{on } \ r = a_0, \tag{7}$$

$$\psi^1 = \psi^{\mathrm{s}} + \psi^{\mathrm{inc}} \quad \text{and} \quad \frac{1}{\rho_1} \frac{\partial \psi^1}{\partial r} = \frac{1}{\rho} \frac{\partial (\psi^{\mathrm{s}} + \psi^{\mathrm{inc}})}{\partial r}, \quad \text{on } \ r = a_1. \tag{8}$$

Solving these boundary conditions (see capsule-boundary-conditions.nb) leads to

$$
\begin{aligned}
T_{nn} = -\frac{J_n(ka_1)}{H_n(ka_1)} - \frac{Y_{\prime}^n(ka_1, ka_1)}{H_n(ka_1)} &\left[Y^n(k_1 a_1, k_1 a_0) J_n'(k_0 a_0) - q_0 J_n(k_0 a_0) Y_{\prime}^n(k_1 a_1, k_1 a_0) \right] \\
\times \big[J_n'(k_0 a_0)(q H_n(ka_1) Y_{\prime}^n(k_1 a_0, k_1 a_1) + H_n'(ka_1) Y^n(k_1 a_1, k_1 a_0)) \\
+ q_0 J_n(k_0 a_0)(q H_n(ka_1) Y_{\prime\prime}^n(k_1 a_1, k_1 a_0) - H_n'(ka_1) Y_{\prime}^n(k_1 a_1, k_1 a_0)) \big]^{-1}. \quad (9)
\end{aligned}
$$

where $q = \rho c/(\rho_1 c_1)$, $q_0 = \rho_0 c_0/(\rho_1 c_1)$, and

$$
\begin{aligned}
Y^n(x, y) &= H_n(x) J_n(y) - H_n(y) J_n(x), & (10) \\
Y_{\prime}^n(x, y) &= H_n(x) J_n'(y) - H_n'(y) J_n(x), & (11) \\
Y_{\prime\prime}^n(x, y) &= H_n'(x) J_n'(y) - H_n'(y) J_n'(x). & (12)
\end{aligned}
$$

1.3 Multiple scattering in 2D

Graf's addition theorem in two spatial dimensions:

$$
H_n(kR_\ell) \mathrm{e}^{\mathrm{i}n\Theta_\ell} = \sum_{m=-\infty}^{\infty} H_{n-m}(kR_{\ell j}) \mathrm{e}^{\mathrm{i}(n-m)\Theta_{\ell j}} J_m(kR_j) \mathrm{e}^{\mathrm{i}m\Theta_j}, \quad \text{for } R_j < R_{\ell j}, \tag{13}
$$

where $(R_{\ell j}, \Theta_{\ell j})$ are the polar coordinates of $\boldsymbol{r}_j - \boldsymbol{r}_\ell$. The above is also valid if we swap H_n for J_n, and swap H_{n-m} for J_{n-m}.

Particle-j scatters a field

$$
u_j = \sum_n f_n^j \mathrm{u}_n(k\boldsymbol{r} - k\boldsymbol{r}_j), \quad \text{for } |\boldsymbol{r} - \boldsymbol{r}_j| > a_j, \tag{14}
$$

where $\boldsymbol{r}_j$ is the centre of particle j.

Let the incident wave, with coordinate system centred at $\boldsymbol{r}_j$, be

$$
u_{\mathrm{inc}} = \sum_n g_n^j \mathrm{v}_n(k\boldsymbol{r} - k\boldsymbol{r}_j), \tag{15}
$$

then the wave exciting particle-j is

$$
u_j^E = \sum_n F_j^n \mathrm{v}_n(k\boldsymbol{r} - k\boldsymbol{r}_j) \tag{16}
$$

where

$$
F_n^j = g_n^j + \sum_{\ell \neq j} \sum_{p=-\infty}^{\infty} f_p^\ell H_{p-m}(kR_{\ell j}) \mathrm{e}^{\mathrm{i}(p-m)\Theta_{\ell j}}. \tag{17}
$$

Using the T-matrix of particle-j we reach $f^j_n = \sum_m T^j_{nm} F^j_m$, which leads to

$$f^j_q = \sum_m T^j_{qm} g^j_m + \sum_{\ell \neq j} \sum_{m,p=-\infty}^{\infty} f^\ell_p T^j_{qm} H_{p-m}(kR_{\ell j}) \mathrm{e}^{\mathrm{i}(p-m)\Theta_{\ell j}}. \tag{18}$$

The above simplifies if we substitute $f^j_q = T^j_{qd} \alpha^j_d$, and then multiple across by $\{T^j_{qn}\}^{-1}$ and sum over q to arrive at

$$\alpha^j_n = g^j_n + \sum_{\ell \neq j} \sum_{m,p=-\infty}^{\infty} H_{p-n}(kR_{\ell j}) \mathrm{e}^{\mathrm{i}(p-n)\Theta_{\ell j}} T^\ell_{pm} \alpha^\ell_m. \tag{19}$$

As a check, if we use (21), then we arrive at equation (2.11) in [3].

In the general formulation below we would have

$$\mathcal{U}_{n'n}(kR_{\ell j}) = H_{n'-n}(kR_{\ell j}) \mathrm{e}^{\mathrm{i}(n'-n)\Theta_{\ell j}}.$$

Note that swapping ℓ for j would result in $\Theta_{\ell j} = \Theta_{j\ell} + \pi$.

2 3D acoustics

For all the details on acoustics in three spatial dimensions see [4]. Here we all only provide:

$$\begin{cases} \mathrm{u}_n(k\boldsymbol{r}) = \mathrm{h}^{(1)}_\ell(kr)\mathrm{Y}_n(\hat{\boldsymbol{r}}), \\ \mathrm{v}_n(k\boldsymbol{r}) = \mathrm{j}_\ell(kr)\mathrm{Y}_n(\hat{\boldsymbol{r}}), \end{cases} \tag{20}$$

where $r = |\boldsymbol{r}|$, $n = \{\ell, m\}$, with summation being over $\ell = 0, 1, 2, 3 \ldots$ and $m = -\ell, -\ell+1, \ldots, -1, 0, 1, \ldots, \ell$, and the spherical Hankel and Bessel functions are denoted $\mathrm{h}^{(1)}_\ell(z)$ and $\mathrm{j}_\ell(z)$, respectively.

3 A sphere

Let ρ and c be the background density and wavespeed, then for a spherical particle with density ρ_j, soundspeed c_j and radius a_j, we have that

$$T_{nq} = -\delta_{nq} \frac{\gamma_j j'_q(ka_j) j_q(k_j a_j) - j_q(ka_j) j'_q(k_j a_j)}{\gamma_j h'_q(ka_j) j_q(k_j a_j) - h_q(ka_j) j'_q(k_j a_j)}, \tag{21}$$

where $\gamma_j = (\rho_j c_j)/(\rho c)$ and $k_j = \omega / c_j$.

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