Multiple scattering of waves

Artur L. Gower[a],

[a] School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL,UK

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Abstract

Here we show and deduce the T-matrix and a general multiple scattering formulation which can be adapted to acoustics, electromagnetism, and elasticity. For details on each specific physical medium see the other documents.

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1 Using a T-matrix

A T-matrix denotes how one single particle scatters waves [5, 4].

For convenience and generality we denote:

$$\begin{aligned} \mathrm{u}_n(k\boldsymbol{r}) &= \text{outgoing spherical waves}, \\ \mathrm{v}_n(k\boldsymbol{r}) &= \text{regular spherical waves}, \end{aligned} \tag{1}$$

where n denotes a multi index which depends on the dimension and if the waves are scalar or vector fields. For example, for scalar waves in three spatial dimensions we have

$$\begin{aligned} \mathrm{u}_n(k\boldsymbol{r}) &= \mathrm{h}_\ell^{(1)}(kr)\mathrm{Y}_n(\hat{\boldsymbol{r}}), \\ \mathrm{v}_n(k\boldsymbol{r}) &= \mathrm{j}_\ell(kr)\mathrm{Y}_n(\hat{\boldsymbol{r}}), \end{aligned} \tag{2}$$

where n is a multi index $n = (\ell, m)$, with $\ell = 0, 1, 2, 3\ldots$ and $m = -\ell, -\ell + 1, \ldots, -1, 0, 1, \ldots, \ell$. Here $\mathrm{h}_\ell^{(1)}(z)$ and $\mathrm{j}_\ell(z)$ are the spherical Hankel and Bessel functions respectively, and Y_n are the spherical harmonic basis functions that are orthonormal with respect to the standard inner product on the unit sphere [2].

Any incident wave and scattered wave[*], centred at the same coordinate axis, can be written as

$$u_{\text{inc}} = \sum_n g_n \mathrm{v}_n(k\boldsymbol{r}), \tag{3}$$

$$u_{\text{sc}} = \sum_n f_n \mathrm{u}_n(k\boldsymbol{r}), \tag{4}$$

for vector waves, such as elastic waves, f_n and $\mathrm{u}_n(k\boldsymbol{r})$ are both vectors for each n, with $f_n \mathrm{u}_n(k\boldsymbol{r})$ being the inner product between these two vectors, the same is true for g_n and $\mathrm{v}_n(k\boldsymbol{r})$.

The T-matrix is an infinite matrix such that

$$f_n = \sum_{n'} T_{nn'} g_{n'}, \tag{5}$$

where for vector waves $T_{nn'}$ is a matrix multiplied with the vector $g_{n'}$. Such a matrix T exists when scattering is a linear operation (elastic scattering).

1.1 Interal field

We can also estimate the field inside the particle by assuming that the field is smooth and continuous. This approach is generally not true for vector wave equations, but is exact for homogeneous spheres and cylinders, but not for a Circular cylindrical capsule.

Assume the field inside the particle can be described by a regular spherical series:

$$v_{\text{in}} = \sum_n b_n \mathrm{v}_n(k_o\boldsymbol{r}), \tag{6}$$

where k_o if the particles wavenumber. Now if we assume that the total field is continuous everywhere so that $u_{\text{inc}} + u_{\text{sc}} = v_{\text{in}}$ on the boundary of the particle. If the field was smooth enough, we could analytically extend the field v_{in} to a spherical boundary, with radius a, which contains the particle. Let's take this as an assumption and equate $u_{\text{inc}} + u_{\text{sc}} = v_{\text{in}}$ for $r = a$. Due to orthogonality of the angular components of the basis functions this will result in

$$g_n \mathrm{v}_n(k\boldsymbol{r}) + f_n \mathrm{u}_n(k\boldsymbol{r}) = b_n \mathrm{v}_n(k_o\boldsymbol{r}), \quad \text{for } |\boldsymbol{r}| = a \tag{7}$$

using the T-matrix we can then write $g_n = T^{-1}_{nm} f_m$, which substituted above leads to

$$b_n = \frac{1}{\mathrm{v}_n(k_o\boldsymbol{r})}[\mathrm{v}_n(k\boldsymbol{r}) T^{-1}_{nm} f_m + \mathrm{u}_n(k\boldsymbol{r}) f_n], \quad \text{for } |\boldsymbol{r}| = a. \tag{8}$$

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[*] For the scattered wave we need only use outgoing spherical waves when measuring the field outside of a sphere which completely encompasses the particle.

2 General multiple scattering

For multiple scattering in higher dimensions and for vector wave equations we use the notation given in [6].

For a point $\boldsymbol{r}$, outside of the circumscribed spheres of all particles, we can write the total field $u(\boldsymbol{r})$ as a sum of the incident wave $u_{\text{inc}}(\boldsymbol{r})$ and all scattered waves in the form [7, 8, 9]

$$u(\boldsymbol{r}) = u_{\text{inc}}(\boldsymbol{r}) + u_{\text{sc}}(\boldsymbol{r}), \quad u_{\text{sc}}(\boldsymbol{r}) = \sum_{i=1}^{N} \sum_{n} f_n^i \mathrm{u}_n(k\boldsymbol{r} - k\boldsymbol{r}_i), \tag{9}$$

where we assumed $|\boldsymbol{r} - \boldsymbol{r}_i| > a_i$ for $i = 1, 2, \dots N$, the f_n^i are coefficients we need to determine.

In general, we can write the multiple scattering system in the form:

$$\alpha_n^i = g_n^i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \sum_{n'n''} \mathcal{U}_{n''n}(k\boldsymbol{r}_i - k\boldsymbol{r}_j) T_{n''n'}^j \alpha_{n'}^j, \tag{10}$$

for $i = 1, 2, \dots, N$, where $f_n^i = \sum_{n'} T_{nn'}^i \alpha_{n'}^i$ and $\mathcal{U}_{nn'}$ is a translation matrix [1, 3]. Let $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{d}$, then the translation matrices for a translation $\boldsymbol{d}$ can be defined by the property [1]

$$\begin{cases} \mathrm{v}_n(k\boldsymbol{r}') = \sum_{n'} \mathcal{V}_{nn'}(k\boldsymbol{d}) \mathrm{v}_{n'}(k\boldsymbol{r}), & \text{for all } \boldsymbol{d} \\ \mathrm{u}_n(k\boldsymbol{r}') = \sum_{n'} \mathcal{V}_{nn'}(k\boldsymbol{d}) \mathrm{u}_{n'}(k\boldsymbol{r}), & |\boldsymbol{r}| > |\boldsymbol{d}| \\ \mathrm{u}_n(k\boldsymbol{r}') = \sum_{n'} \mathcal{U}_{nn'}(k\boldsymbol{d}) \mathrm{v}_{n'}(k\boldsymbol{r}), & |\boldsymbol{r}| < |\boldsymbol{d}| \end{cases} \tag{11}$$

The coefficients $g_{n'}^i$ depend on the form of the incident wave. If we can use the representation (3) then we have that

$$g_n^i = \sum_{n'} \mathcal{V}_{n'n}(\boldsymbol{r}_i) g_{n'}.$$

2.1 Turning equations into code

For easy implementation we need the functions:

$$\psi_{\text{inc}} \mapsto g_n^j \quad \text{and} \quad \text{particle} \mapsto T_{nn'}^j.$$

For efficient implementation we rewrite (10) as a matrix equation. Let

$$(\boldsymbol{\alpha}_j)_n = \alpha_n^j, \quad (\boldsymbol{g}_j)_n = g_n^j, \tag{12}$$

$$(\boldsymbol{T}_j)_{nn'} = T_{nn'}^j, \quad (\boldsymbol{\mathcal{U}}_{j\ell})_{n'n} = \mathcal{U}_{n'n}(k\boldsymbol{r}_j - k\boldsymbol{r}_\ell), \tag{13}$$

Then

$$\sum_\ell (\delta_{j\ell} + (\delta_{j\ell} - 1)\boldsymbol{\mathcal{U}}_{j\ell}^{\mathrm{T}} \boldsymbol{T}_\ell)\boldsymbol{\alpha}_\ell = \boldsymbol{g}_j, \tag{14}$$

where $\cdot^{\mathrm{T}}$ is the transpose operation. The above then leads to a block matrix equation:

$$\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{\mathcal{U}}_{12}^{\mathrm{T}}\boldsymbol{T}_2 & \cdots & -\boldsymbol{\mathcal{U}}_{1(N-1)}^{\mathrm{T}}\boldsymbol{T}_{N-1} & -\boldsymbol{\mathcal{U}}_{1N}^{\mathrm{T}}\boldsymbol{T}_N \\ -\boldsymbol{\mathcal{U}}_{21}^{\mathrm{T}}\boldsymbol{T}_1 & \boldsymbol{I} & -\boldsymbol{\mathcal{U}}_{23}^{\mathrm{T}}\boldsymbol{T}_3 & \cdots & -\boldsymbol{\mathcal{U}}_{2N}^{\mathrm{T}}\boldsymbol{T}_N \\ & \vdots & & & \vdots \\ -\boldsymbol{\mathcal{U}}_{N1}^{\mathrm{T}}\boldsymbol{T}_1 & \cdots & \cdots & -\boldsymbol{\mathcal{U}}_{N(N-1)}^{\mathrm{T}}\boldsymbol{T}_{N-1} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}_1 \\ \vdots \\ \boldsymbol{g}_N \end{bmatrix} \tag{15}$$

3 Periodic multiple scattering

Here we consider a unit cell filled with particles that is repeated periodically. The particles can take any positions within the cell.

Let us start with the simplest case of identical particles centered at the positions $\boldsymbol{r}_1 \in \mathcal{P}$, where $\mathcal{P}$ is some countable set of vectors we define later.

The total field is again given by (9). However, if we assume the source is periodic with

$$u_{\mathrm{inc}}(\boldsymbol{r}) = u_{\mathrm{inc}}(\boldsymbol{r} + \boldsymbol{r}_1), \quad \text{for every } \ \boldsymbol{r}_1 \in \mathcal{P}, \tag{16}$$

then, due to symmetry, the scattering coefficients of every particles is the same $f_n := f_n^i$, and as a result the total field is given by

$$u(\boldsymbol{r}) = u_{\mathrm{inc}}(\boldsymbol{r}) + \sum_n f_n \sum_i \mathrm{u}_n(k\boldsymbol{r} - k\boldsymbol{r}_i).$$

Taking $\boldsymbol{r} = \boldsymbol{v} + \boldsymbol{r}_j$, we can then write the wave arriving at (or exciting) the particle at $\boldsymbol{r}_j$ in the form

$$u_{\mathrm{ex}}^j(\boldsymbol{v}) = u_{\mathrm{inc}}(\boldsymbol{v}) + \sum_n f_n \sum_{i \neq j} \mathrm{u}_n(k\boldsymbol{v} + k\boldsymbol{r}_j - k\boldsymbol{r}_i),$$

where we used (16). Now we assume that $\boldsymbol{v}$ is close to the boundary of particle in the unit cell (which is needed to apply boundary conditions), so that $|\boldsymbol{v}| < |\boldsymbol{r}_j - \boldsymbol{r}_i|$ for $j \neq i$, which allows us to use $(11)_3$ to write the above as a series of regular spherical waves centred at $\boldsymbol{r}_j$ in the form

$$u^j_{\text{ex}}(\boldsymbol{v}) = \sum_{n_1} g_{n_1} \mathrm{v}_{n_1}(\boldsymbol{r}) + \sum_n f_n \sum_{i \neq j} \sum_{n_1} \mathcal{U}_{nn_1}(k\boldsymbol{r}_j - k\boldsymbol{r}_i) \mathrm{v}_{n_1}(k\boldsymbol{v}).$$

Using the T-matrix formulation, we can now link the scattering coefficients f_n to the coefficients of the regular wave above to get

$$f_{n'} = \sum_{n_1} T_{n'n_1} g_{n_1} + \sum_{nn_1} f_n \sum_{i \neq j} T_{n'n_1} \mathcal{U}_{nn_1}(k\boldsymbol{r}_j - k\boldsymbol{r}_i), \tag{17}$$

which can be solved for f_n. The main issue is how to truncate the series $\sum_j \mathcal{U}_{nn_1}(k\boldsymbol{r}_j)$ in j, but I think this would work quite well.

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