Notes on the T-matrix in 2D

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Abstract

short explanation

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1 Single scatterer

We will be following a similar notation as used in A T-Matrix Reduced Order Model Software [2, 1].

Any incident and scattered wave in 2D, centred at the same polar coordinate axis, can be written as

$$\psi^{\text{inc}} = \sum_{n=-\infty}^{\infty} f_n J_n(kr) \mathrm{e}^{\mathrm{i}n\theta}, \tag{1}$$

$$\psi^{\text{s}} = \sum_{n=-\infty}^{\infty} a_n H_n(kr) \mathrm{e}^{\mathrm{i}n\theta}. \tag{2}$$

The T-matrix is an infinite matrix such that

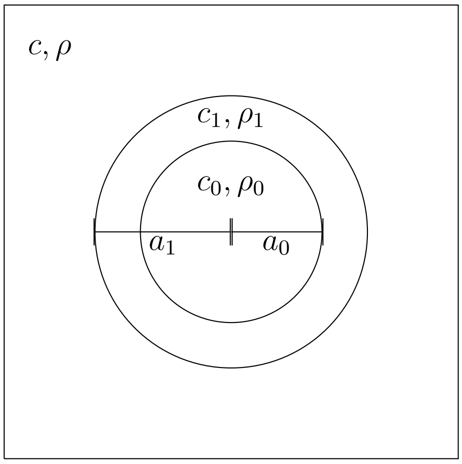
$$a_n = \sum_{m=-\infty}^{\infty} T_{nm} f_m. \tag{3}$$

Such a matrix T exists when scattering is a linear operation (elastic scattering).

For instance, if ρ and c are the background density and wavespeed, then for a circular scatterer with density ρ_j, soundspeed c_j and radius a_j, we have that

$$T_{nm} = -\delta_{nm} Z_j^m, \quad \text{with} \quad Z_j^m = \frac{q_j J_m'(ka_j) J_m(k_j a_j) - J_m(ka_j) J_m'(k_j a_j)}{q_j H_m'(ka_j) J_m(k_j a_j) - H_m(ka_j) J_m'(k_j a_j)}, \tag{4}$$

where $q_j = (\rho_j c_j)/(\rho c)$ and $k_j = \omega / c_j$.



1.1 Single circular capsule

$$\psi^0 = \sum_{n=-\infty}^{\infty} f_n^0 J_n(k_0 r) \mathrm{e}^{in\theta}, \tag{5}$$

$$\psi^1 = \sum_{n=-\infty}^{\infty} \left[f_n^1 J_n(k_1 r) + a_n^1 H_n(k_1 r) \right] \mathrm{e}^{in\theta}. \tag{6}$$

Applying the boundary conditions,

$$\psi^0 = \psi^1 \quad \text{and} \quad \frac{1}{\rho_0}\frac{\partial \psi^0}{\partial r} = \frac{1}{\rho_1}\frac{\partial \psi^1}{\partial r}, \quad \text{on} \;\; r = r_0, \tag{7}$$

$$\psi^1 = \psi^{\mathrm{s}} + \psi^{\mathrm{inc}} \quad \text{and} \quad \frac{1}{\rho_1}\frac{\partial \psi^1}{\partial r} = \frac{1}{\rho}\frac{\partial (\psi^{\mathrm{s}} + \psi^{\mathrm{inc}})}{\partial r}, \quad \text{on} \;\; r = r_1. \tag{8}$$

Solving these boundary conditions (see capsule-boundary-conditions.nb) leads to

$$\begin{aligned} T_{nn} = &-\frac{J_n(ka_1)}{H_n(ka_1)} - \frac{Y\prime^n(ka_1, ka_1)}{H_n(ka_1)} \left[Y^n(k_1a_1, k_1a_0) J_n'(k_0a_0) - q_0 J_n(k_0a_0) Y\prime^n(k_1a_1, k_1a_0) \right] \\ &\times \big[J_n'(k_0a_0)(qH_n(ka_1) Y\prime^n(k_1a_0, k_1a_1) + H_n'(ka_1) Y^n(k_1a_1, k_1a_0)) \\ &+ q_0 J_n(k_0a_0)(qH_n(ka_1) Y\prime\prime^n(k_1a_1, k_1a_0) - H_n'(ka_1) Y\prime^n(k_1a_1, k_1a_0)) \big]^{-1}. \end{aligned} \tag{9}$$

where $q = \rho c/(\rho_1 c_1)$, $q_0 = \rho_0 c_0/(\rho_1 c_1)$, and

$$Y^n(x, y) = H_n(x) J_n(y) - H_n(y) J_n(x), \tag{10}$$

$$Y\prime^n(x, y) = H_n(x) J_n'(y) - H_n'(y) J_n(x), \tag{11}$$

$$Y\prime\prime^n(x, y) = H_n'(x) J_n'(y) - H_n'(y) J_n'(x). \tag{12}$$

2 Multiple scattering

Graf's addition theorem

$$H_n(kR_\ell)\mathrm{e}^{\mathrm{i}n\Theta_\ell} = \sum_{m=-\infty}^{\infty} H_{n-m}(kR_{\ell j})\mathrm{e}^{\mathrm{i}(n-m)\Theta_{\ell j}} J_m(kR_j)\mathrm{e}^{\mathrm{i}m\Theta_j}, \quad \text{for} \;\; R_j < R_{\ell j}, \tag{13}$$

where $(R_{\ell j}, \Theta_{\ell j})$ are the polar coordinates of $\boldsymbol{x}_j - \boldsymbol{x}_\ell$. The above is also valid if we swap H_n for J_n, and swap H_{n-m} for J_{n-m}.

Particle-j scatters a field

$$\psi_j = \sum_{m=-\infty}^{\infty} A_j^m H_m(kR_j)\mathrm{e}^{\mathrm{i}m\Theta_j}, \quad \text{for} \;\; R_j > a_j, \tag{14}$$

where (R_j, Θ_j) are the polar coordinates of $\boldsymbol{x} - \boldsymbol{x}_j$, where $\boldsymbol{x}_j$ is the centre of particle j.

Let the incident wave, with coordinate system centred at $\boldsymbol{x}_j$, be

$$\psi_{\text{inc}} = \sum_{m=-\infty}^{\infty} f_j^m J_m(kR_j)\mathrm{e}^{\mathrm{i}m\Theta_j}, \tag{15}$$

then the wave exciting particle-j is

$$\psi_j^E = \sum_{m=-\infty}^{\infty} F_j^m J_m(kR_j)\mathrm{e}^{\mathrm{i}m\Theta_j}, \tag{16}$$

where

$$F_j^m = f_j^m + \sum_{\ell\neq j} \sum_{p=-\infty}^{\infty} A_\ell^p H_{p-m}(kR_{\ell j})\mathrm{e}^{\mathrm{i}(p-m)\Theta_{\ell j}}. \tag{17}$$

Using the T-matrix of particle-j we reach $A_j^n = \sum_m T_j^{nm} F_j^m$, which leads to

$$A_j^q = \sum_{m=-\infty}^{\infty} T_j^{qm} f_j^m + \sum_{\ell\neq j} \sum_{m,p=-\infty}^{\infty} A_\ell^p T_j^{qm} H_{p-m}(kR_{\ell j})\mathrm{e}^{\mathrm{i}(p-m)\Theta_{\ell j}}. \tag{18}$$

The above simplifies if we substitute $A_j^q = T_j^{qd}\alpha_j^d$, and then multiple across by $\{T_j^{-1}\}^{qn}$ and sum over q to arrive at

$$\alpha_j^n = f_j^n + \sum_{\ell\neq j} \sum_{m,p=-\infty}^{\infty} H_{p-n}(kR_{\ell j})\mathrm{e}^{\mathrm{i}(p-n)\Theta_{\ell j}} T_\ell^{pm}\alpha_\ell^m. \tag{19}$$

As a check, if we use (4), then we arrive at equation (2.11) in [3]. For easy implementation we need the functions:

$$\psi_{\text{inc}} \mapsto f_j^m \quad \text{and} \quad \text{particle} \mapsto T_j^{nm}.$$

For efficient implementation we rewrite (19) as a matrix equation. Let

$$(\boldsymbol{\alpha}_j)_n = \alpha_j^n, \quad (\boldsymbol{f}_j)_n = f_j^n, \tag{20}$$

$$(\boldsymbol{T}_j)_{nm} = T_j^{nm}, \quad (\boldsymbol{\Psi}_{j\ell})_{np} = H_{p-n}(kR_{\ell j})\mathrm{e}^{\mathrm{i}(p-n)\Theta_{\ell j}}, \tag{21}$$

and note that changing the order of ℓ and j, makes $\Theta_{\ell j} = \Theta_{j\ell} + \pi$. Then

$$\sum_{\ell}(\delta_{j\ell} + (\delta_{j\ell} - 1)\boldsymbol{\Psi}_{j\ell}\boldsymbol{T}_\ell)\boldsymbol{\alpha}_\ell = \boldsymbol{f}_j, \tag{22}$$

which leads to one massive square matrix:

$$\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{\Psi}_{12}\boldsymbol{T}_2 & \cdots & -\boldsymbol{\Psi}_{1(N-1)}\boldsymbol{T}_{N-1} & -\boldsymbol{\Psi}_{1N}\boldsymbol{T}_N \\ -\boldsymbol{\Psi}_{21}\boldsymbol{T}_1 & \boldsymbol{I} & -\boldsymbol{\Psi}_{23}\boldsymbol{T}_3 & \cdots & -\boldsymbol{\Psi}_{2N}\boldsymbol{T}_N \\ & \vdots & & & \vdots \\ -\boldsymbol{\Psi}_{N1}\boldsymbol{T}_1 & \cdots & \cdots & -\boldsymbol{\Psi}_{N(N-1)}\boldsymbol{T}_{N-1} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \vdots \\ \boldsymbol{f}_N \end{bmatrix} \tag{23}$$

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